EXERCISES [MAI 3.15]

VECTORS AND KINEMATICS

SOLUTIONS

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A. Paper 1 questions (SHORT)

1. (a) (3,5) and (11,11)

(b) velocity $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, speed $|v| = 5$

(c) $11\sqrt{2}$

(d) 10

2.

Cases	Vector **AB**	Velocity vector	Equation of motion
at point B(8,6) after 1 sec	$\begin{pmatrix} 6 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 3 \end{pmatrix}$	$\vec{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 6 \\ 3 \end{pmatrix}$
at point B(8,6) after 3 sec	$\begin{pmatrix} 6 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\vec{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

3. (a) $\boldsymbol{v_1} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\boldsymbol{v_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ speeds: $|\boldsymbol{v_1}| = \sqrt{9} = 3$, $|\boldsymbol{v_2}| = \sqrt{9} = 3$

(b) (5,6,7) satisfies $\boldsymbol{r_1}$ for $t = 2$ and $\boldsymbol{r_2}$ for $t = 1$

(c) They do not collide since at the point of intersection $t_1 \neq t_2$.

4. (a) (4,9,7)

(b) $t_1 = 2$ while $t_2 = 1$

(c) $|v_1| = \sqrt{21}$, $|v_2| = \sqrt{13}$

(d) $d = \sqrt{21}$.

5. (a) $\sqrt{16+9} = \sqrt{25} = 5$

(b) $\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (not unique)

(b) $\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$ (so B is (6, 7))

6. (a) (i) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$ (ii) speed $= \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50}$ $(= 5\sqrt{2})$

(b) $\boldsymbol{r} = \begin{pmatrix} 6 \\ -2 \\ 10 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$

7. (a) $p = 2 \Rightarrow \begin{pmatrix} 0 \\ 12 \end{pmatrix} + 2\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$

(b) (i) equating components

$0 + 5p = 14 + q$,

$12 - 3p = 0 + 3q$

$\Rightarrow p = 3,\ q = 1$

(ii) The coordinates of P are (15, 3)

(c) No, they do not start at the same time since $p \neq q$. Car 1 starts two seconds earlier.

8.

Equation of motion	Velocity $\vec{v}$ (in terms of t)	Acceleration $\vec{a}$ (in terms of t)
$\vec{r} = \begin{pmatrix} 3t+2 \\ 5t+1 \end{pmatrix}$	$\vec{v} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$	$\vec{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\vec{r} = \begin{pmatrix} 3+t \\ 2-t \end{pmatrix}$	$\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\vec{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\vec{r} = \begin{pmatrix} 3t+7 \\ 2t^2 - t + 1 \end{pmatrix}$	$\vec{v} = \begin{pmatrix} 3 \\ 4t-1 \end{pmatrix}$	$\vec{a} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$
$\vec{r} = \begin{pmatrix} t^2+1 \\ t^3+1 \end{pmatrix}$	$\vec{v} = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$	$\vec{a} = \begin{pmatrix} 2 \\ 6t \end{pmatrix}$
$\vec{r} = \begin{pmatrix} 5t^2 \\ 2\sin t \end{pmatrix}$	$\vec{v} = \begin{pmatrix} 10t \\ 2\cos t \end{pmatrix}$	$\vec{a} = \begin{pmatrix} 10 \\ -2\sin t \end{pmatrix}$

9. (a) $\vec{v} = \begin{pmatrix} -5\pi \sin \dfrac{\pi}{2}t \\ 5\pi \cos \dfrac{\pi}{2}t \end{pmatrix}$, $\vec{a} = \begin{pmatrix} -\dfrac{5\pi^2}{2}\cos \dfrac{\pi}{2}t \\ -\dfrac{5\pi^2}{2}\sin \dfrac{\pi}{2}t \end{pmatrix}$

For $t = 0$, $\vec{r} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 5\pi \end{pmatrix}$, $\vec{a} = \begin{pmatrix} -\dfrac{5\pi^2}{2} \\ 0 \end{pmatrix}$

(b) speed $= |\vec{v}| = \sqrt{\left(-5\pi \sin \dfrac{\pi}{2}t\right)^2 + \left(5\pi \cos \dfrac{\pi}{2}t\right)^2} = \sqrt{25\pi^2\left(\sin^2 \dfrac{\pi}{2}t + \cos^2 \dfrac{\pi}{2}t\right)} = 5\pi$

(c) $x^2 + y^2 = 100$

B. Paper 2 questions (LONG)

10. (a) (i) $\boldsymbol{r}_1 = \begin{bmatrix}16\\12\end{bmatrix} + t\begin{bmatrix}12\\-5\end{bmatrix}$ $t = 0 \Rightarrow \boldsymbol{r}_1 = \begin{bmatrix}16\\12\end{bmatrix}$ $|\boldsymbol{r}_1| = \sqrt{(16^2 + 12^2)} = 20$

(ii) Velocity vector $= \begin{bmatrix}12\\-5\end{bmatrix} \Rightarrow$ speed $= \sqrt{(12^2 + (-5)^2)} = 13$

(b) $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}16\\12\end{bmatrix} + t\begin{bmatrix}12\\-5\end{bmatrix} \Rightarrow \dfrac{x-16}{12} = \dfrac{y-12}{-5} \Rightarrow 5x - 80 = 144 - 12y \Rightarrow 5x + 12y = 224$

(c) $\boldsymbol{v}_1 = \begin{bmatrix}12\\-5\end{bmatrix}$ $\boldsymbol{v}_2 = \begin{bmatrix}2.5\\6\end{bmatrix}$, $\boldsymbol{v}_1.\boldsymbol{v}_2 = \begin{bmatrix}12\\-5\end{bmatrix}.\begin{bmatrix}2.5\\6\end{bmatrix} = 30 - 30 \Rightarrow \boldsymbol{v}_1.\boldsymbol{v}_2 = 0 \Rightarrow \theta = 90°$

(d) (i) $\dfrac{x-23}{2.5} = \dfrac{y+5}{6} \Rightarrow 6x - 138 = 2.5y + 12.5 \Rightarrow 12x - 276 = 5y + 25 \Rightarrow 12x - 5y = 301$

(ii) $\left.\begin{matrix}5x + 12y = 224\\12x - 5y = 301\end{matrix}\right\} \Rightarrow \left.\begin{matrix}25x + 60y = 1120\\144x - 60y = 3612\end{matrix}\right\} \Rightarrow (x, y) = (28, 7)$

(e) $16 + 12t = 23 + 2.5t \Rightarrow 9.5t = 7$

$12 - 5t = -5 + 6t \quad \Rightarrow 17 = 11t$

$\dfrac{7}{9.5} \neq \dfrac{17}{11} \Rightarrow$ planes cannot be at the same place at the same time

OR

$\boldsymbol{r}_1 = \begin{bmatrix}28\\7\end{bmatrix} \Leftrightarrow \begin{bmatrix}28\\7\end{bmatrix} = \begin{bmatrix}16\\12\end{bmatrix} + t\begin{bmatrix}12\\-5\end{bmatrix} \Leftrightarrow t = 1$

$\boldsymbol{r}_2 = \begin{bmatrix}28\\7\end{bmatrix} \Leftrightarrow \begin{bmatrix}28\\7\end{bmatrix} = \begin{bmatrix}23\\-5\end{bmatrix} + t\begin{bmatrix}2.5\\6\end{bmatrix} \Leftrightarrow t = 2$

11. (a) At $t = 2$, $\begin{pmatrix}2\\0\end{pmatrix} + 2\begin{pmatrix}0.7\\1\end{pmatrix} = \begin{pmatrix}3.4\\2\end{pmatrix}$

Distance from (0, 0) = $\sqrt{3.4^2 + 2^2} = 3.94$ m

(b) $\left|\begin{pmatrix}0.7\\1\end{pmatrix}\right| = \sqrt{0.7^2 + 1^2} = 1.22$ m s^{-1}

(c) $x = 2 + 0.7\,t$ and $y = t$

$x - 0.7y = 2$

(d) $y = 0.6x + 2$ and $x - 0.7y = 2$

$x = 5.86$ and $y = 5.52$ $\left(\text{or } x = \dfrac{170}{29} \text{ and } y = \dfrac{160}{29}\right)$

(e) The time of the collision may be found by solving

$\begin{pmatrix}5.86\\5.52\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0.7\\1\end{pmatrix}t \Rightarrow t = 5.52$ s

[*ie* collision occurred 5.52 seconds after the vehicles set out].

Distance d travelled by the motorcycle is given by

$d = \left|\begin{pmatrix}5.86\\5.52\end{pmatrix} - \begin{pmatrix}0\\2\end{pmatrix}\right| = \sqrt{(5.86)^2 + (3.52)^2} = \sqrt{46.73} = 6.84$ m

Speed of the motorcycle $= \dfrac{d}{t} = \dfrac{6.84}{5.52} = 1.24$ m s^{-1}

12. (a) speed= $\sqrt{3^2+4^2+10^2}=\sqrt{125}=5\sqrt{5}$, 11.2, (metres per minute)

(b) Velocity vector: $\begin{pmatrix}3\\16\\39\end{pmatrix}-\begin{pmatrix}-5\\10\\23\end{pmatrix}$ Dividing by 2 to give $\begin{pmatrix}4\\3\\8\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-5\\10\\23\end{pmatrix}+t\begin{pmatrix}4\\3\\8\end{pmatrix}$$

(c) (i) At Q, $\begin{pmatrix}3\\2\\7\end{pmatrix}+t\begin{pmatrix}3\\4\\10\end{pmatrix}=\begin{pmatrix}-5\\10\\23\end{pmatrix}+t\begin{pmatrix}4\\3\\8\end{pmatrix}\Rightarrow 3+3t=-5+4t\Rightarrow t=8$

after 8 minutes, 13:08

(ii) Substituting for *t*: $x=27, y=34, z=87$ i.e. (27, 34, 87)

(d) direction vectors $\boldsymbol{d}_1=\begin{pmatrix}3\\4\\10\end{pmatrix}$ and $\boldsymbol{d}_2=\begin{pmatrix}4\\3\\8\end{pmatrix}$ $\boldsymbol{d}_1\bullet\boldsymbol{d}_2=104$

$\cos\theta=\dfrac{104}{\sqrt{125}\sqrt{89}}=0.98601\ldots\theta=0.167$(radians) (accept $\theta=9.59°$)

13. (a) (i) $\vec{AB}=\begin{pmatrix}200\\400\end{pmatrix}-\begin{pmatrix}-600\\-200\end{pmatrix}=\begin{pmatrix}800\\600\end{pmatrix}$

(ii) $|\vec{AB}|=\sqrt{800^2+600^2}=1000$ unit vector $=\dfrac{1}{1000}\begin{pmatrix}800\\600\end{pmatrix}=\begin{pmatrix}0.8\\0.6\end{pmatrix}$

(b) (i) $\boldsymbol{v}=250\begin{pmatrix}0.8\\0.6\end{pmatrix}=\begin{pmatrix}200\\150\end{pmatrix}$

(ii) at 13:00, $t=1$: $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-600\\-200\end{pmatrix}+1\begin{pmatrix}200\\150\end{pmatrix}=\begin{pmatrix}-400\\-50\end{pmatrix}$

(iii) $|\vec{AB}|=1\,000$

Time $=\dfrac{1\,000}{250}=4$ (hours)

over town B at 16:00 (4 pm, 4:00 pm)

(c) ***Note:*** *There are a variety of approaches. The table shows some of them*

Time for A to B to C = 9 hours	Distance from A to B to C = 2250 km	Fuel used from A to B $=1800\times4=7200$ litres
Light goes on after 16000 litres	Light goes on after 16000 litres	Fuel remaining = 9800 litres
Time for 16 000 litres $=\dfrac{16000}{1800}$ $(=8.889)$ Time remaining is $=\dfrac{1}{9}(=0.111)$ hour	Distance on 16000 litres $=\dfrac{16000}{1800}\times250$ $(=2222.22)$ km	Hours before light $\dfrac{8800}{1800}(=4.889)$ Time remaining is $=\dfrac{1}{9}(=0.111)$ hour
Distance $=\dfrac{1}{9}\times250$ = 27.8 km	Distance to C = 2250 – 2222.22 = 27.8 km	Distance $=\dfrac{1}{9}\times250$ = 27.8 km

14. (a) At 13:00, $t = 1 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + 1 \times \begin{pmatrix} 6 \\ -8 \end{pmatrix} = \begin{pmatrix} 6 \\ 20 \end{pmatrix}$

(b) (i) Velocity vector: $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ (km h^{-1}) (ii)Speed = $\sqrt{(6^2 + (-8)^2)}$;= 10; 10 km h^{-1}

(c) $\left.\begin{array}{l} x = 6t \\ y = 28 - 8t \end{array}\right\} \Rightarrow \frac{x}{6} = \frac{y - 28}{-8} \Rightarrow 4x + 3y = 84$

(d) They collide if $\begin{pmatrix} 18 \\ 4 \end{pmatrix}$ lies on path;

EITHER (18, 4) lies on $4x + 3y = 84$

$\Leftrightarrow 4 \times 18 + 3 \times 4 = 84 \Leftrightarrow 72 + 12 = 84$; OK;

$x = 18 \Rightarrow 18 = 6t \Rightarrow t = 3$, collide at 15:00

OR $\begin{pmatrix} 18 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + t\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ for some t, $\Leftrightarrow \left\{\begin{array}{r} 18 = 6t \\ \textbf{and } 4 = 28 - 8t \end{array}\right\} \Leftrightarrow \left\{\begin{array}{r} t = 3 \\ \textbf{and } t = 3 \end{array}\right\}$

They collide at 15:00

(e) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 4 \end{pmatrix} + (t-1)\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 18 + 5t - 5 \\ 4 + 12t - 12 \end{pmatrix} = \begin{pmatrix} 13 \\ -8 \end{pmatrix} + t\begin{pmatrix} 5 \\ 12 \end{pmatrix}$

(f) At $t = 3$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 + 3 \times 5 \\ -8 + 3 \times 12 \end{pmatrix} = \begin{pmatrix} 28 \\ 28 \end{pmatrix}$

distance = $\sqrt{(10^2 + 24^2)} = \sqrt{(676)}$ = 26 km apart

15. (a) (i) $\overrightarrow{OA} = \begin{pmatrix} 240 \\ 70 \end{pmatrix}$ OA = $\sqrt{240^2 + 70^2}$ = 250

unit vector = $\frac{1}{250}\begin{pmatrix} 240 \\ 70 \end{pmatrix} = \begin{pmatrix} 0.96 \\ 0.28 \end{pmatrix}$

(ii) $\boldsymbol{v} = 300\begin{pmatrix} 0.96 \\ 0.28 \end{pmatrix} = \begin{pmatrix} 288 \\ 84 \end{pmatrix}$

(iii) $t = \frac{240}{288} = \frac{5}{6}$ hr (= 50 min)

(b) $\overrightarrow{AB} = \begin{pmatrix} 480 - 240 \\ 250 - 70 \end{pmatrix} = \begin{pmatrix} 240 \\ 180 \end{pmatrix}$ AB = $\sqrt{240^2 + 180^2}$ = 300

$\cos\theta = \frac{\overrightarrow{OA} \bullet \overrightarrow{AB}}{OA \times AB} = \frac{(240)(240) + (70)(180)}{(250)(300)} = 0.936 \Rightarrow \theta = 20.6°$

(c) (i) $\overrightarrow{AX} = \begin{pmatrix} 339 - 240 \\ 238 - 70 \end{pmatrix} = \begin{pmatrix} 99 \\ 168 \end{pmatrix}$

(ii) $\begin{pmatrix} -3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 240 \\ 180 \end{pmatrix} = -720 + 720 = 0$

$\Rightarrow \boldsymbol{n} \perp \overrightarrow{AB}$

(iii) Projection of $\overrightarrow{AX}$ in the direction of $\boldsymbol{n}$ is

XY = $\frac{1}{5}\begin{pmatrix} 99 \\ 168 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \frac{-297 + 672}{5}$ = 75

(d) AX = $\sqrt{99^2 + 168^2}$ = 195

AY = $\sqrt{195^2 - 75^2}$ = 180 km

16. (a) $\left|\begin{pmatrix}18\\24\end{pmatrix}\right| = 30 \text{ km h}^{-1}$ $\quad \left|\begin{pmatrix}36\\-16\end{pmatrix}\right| = \sqrt{36^2 + (-16)^2} = 39.4$

(b) (i) After ½ hour, position vectors are $\begin{pmatrix}9\\12\end{pmatrix}$ and $\begin{pmatrix}18\\-8\end{pmatrix}$

(ii) At 6.30 am, vector joining their positions is

$\begin{pmatrix}9\\12\end{pmatrix} - \begin{pmatrix}18\\-8\end{pmatrix} = \begin{pmatrix}-9\\20\end{pmatrix}$ (or $\begin{pmatrix}9\\-20\end{pmatrix}$)

$\left|\begin{pmatrix}-9\\20\end{pmatrix}\right| = \sqrt{481}$ (= 21.9 km to 3 sf)

(c) The Toyundai must continue until its position vector is $\begin{pmatrix}18\\k\end{pmatrix}$

Clearly k = 24, *ie* position vector $\begin{pmatrix}18\\24\end{pmatrix}$.

To reach this position, it must travel for 1 hour in total.
Hence the crew starts work at 7.00 am

(d) Southern (Chryssault) crew lays 800 × 5 = 4000 m
Northern (Toyundai) crew lays 800 × 4.5 = 3600 m
Total by 11.30 am = 7.6 km
Their starting points were 24 – (–8) = 32 km apart
Hence they are now 32 – 7.6 = 24.4 km apart

(e) Position vector of Northern crew at 11.30 am is $\begin{pmatrix}18\\24-3.6\end{pmatrix} = \begin{pmatrix}18\\20.4\end{pmatrix}$

Distance to base camp = $\left|\begin{pmatrix}18\\20.4\end{pmatrix}\right|$ = 27.2 km

Time to cover this distance = $\frac{27.2}{30} \times 60$ = 54.4 min = 54 minutes (to the nearest minute)

17. (a) unit vector (e_b) $= \frac{1}{\sqrt{3^2+4^2+0^2}}\begin{pmatrix}3\\4\\0\end{pmatrix} = \begin{pmatrix}0.6\\0.8\\0\end{pmatrix}$, direction vector for $\boldsymbol{b}$, $\boldsymbol{v_b} = 18\boldsymbol{e_b} = \begin{pmatrix}10.8\\14.4\\0\end{pmatrix}$

$\boldsymbol{b} = \boldsymbol{b}_0 + t\boldsymbol{v_b} = \begin{pmatrix}0\\0\\5\end{pmatrix} + t\begin{pmatrix}10.8\\14.4\\0\end{pmatrix}$

(b) (i) $t = 0 \Rightarrow (49, 32, 0)$

(ii) $v_h = \sqrt{(-48)^2 + (-24)^2 + 6^2} = 54(\text{km h}^{-1})$

(c) (i) At R, $\begin{pmatrix}10.8t\\14.4t\\5\end{pmatrix} = \begin{pmatrix}49-48t\\32-24t\\6t\end{pmatrix}$

$t = \frac{5}{6}$ (= 0.833) (hours)

(ii) For $t = \frac{5}{6}$ into expression for b or h, (9,12,5)